Ergodic Theory
Solutions to Problem Sheet 5

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P1. Show that every irrational number has a continued fraction expansion and that this representation is unique.

> Existence: We may assume that $x \in (0,1)$ is an irrational number. Let T be the Gauss map and define a sequence (a_n), by
>
> $$a_n = a_n(x) = \left\lfloor \frac{1}{T^{n-1}x} \right\rfloor .$$
>
> We show that
>
> $$x = [0; a_1, a_2, \dots].$$
>
> Let $u = [0; a_1, a_2, \dots]$ be the limit of the $n-$th convergent $a_n = p_n/q_n$ (where p_n, q_n are coprime), as in Proposition 72 of the lecture notes. For the existence, it is enough to show that $x = u$. We claim that
>
> $$[0; a_1, \dots, a_{2n}] = \frac{p_{2n}}{q_{2n}} < x < \frac{p_{2n+1}}{q_{2n+1}} = [0; a_1, \dots, a_{2n+1}]. \tag{1}$$
>
> Assuming the claim, by taking limits in (1), we obtain that
>
> $$u = \lim_{n\to\infty} \frac{p_{2n}}{q_{2n}} \le x \le \lim_{n\to\infty} \frac{p_{2n+1}}{q_{2n+1}} = u,$$
>
> hence $u = x$.
>
> We prove the claim by induction on n. Recall that $p_0/q_0 = 0$ and $p_1/q_1 = 1/a_1$, so (1) holds for $n = 0$, by the definition of (a_n). Now assume that (1) holds for a given $n \ge 0$. Apply the induction hypothesis to $T(x)$. Notice that
>
> $$a_n(Tx) = \left\lfloor \frac{1}{T^{n-1}(Tx)} \right\rfloor = \left\lfloor \frac{1}{T^{n}x} \right\rfloor = a_{n+1}(x).$$
>
> Thus, if we apply the induction hypothesis to $T(x)$ and recall that T acts as the left shift on the continued fraction representation of a number, we obtain that
>
> $$[0; a_2, \dots, a_{2n+1}] < T(x) < [0; a_2, \dots, a_{2n+2}].$$
>
> This implies that
>
> $$[a_1; a_2, \dots, a_{2n+1}] < \frac{1}{x} < [a_1; a_2, \dots, a_{2n+2}],$$

since $T(x) = 1/x - a_1$. Therefore,

$$[0; a_1, \dots, a_{2n+2}] = \frac{1}{[a_1; a_2, \dots, a_{2n+2}]} < x < \frac{1}{[a_1; a_2, \dots, a_{2n+1}]} = [0; a_1, \dots, a_{2n+1}].$$

Applying the induction hypothesis again to $T^2(x)$ and using that $T^2(x) = \frac{1}{T(x)} - a_2$, we get exactly what we want.

Uniqueness: Here we prove the more general fact that the map that sends the sequence $(a_0, a_1, \dots) \in \mathbb{Z} \times \mathbb{N}^{\mathbb{N}}$ to the limit of the $n-$th congruent is injective.

Let $(a_0, a_1, \dots) \in \mathbb{Z} \times \mathbb{N}^{\mathbb{N}}$ be given and $u = [a_0; a_1, \dots]$. Then notice that

$$u = a_0 + \frac{1}{[a_1; a_2, \dots]}$$

and so

$$u \in (a_0, a_0 + 1/a_1) \subset (a_0, a_0 + 1),$$

showing that u uniquely determines a_0.

Now let

$$u' := \frac{1}{u - a_0} = [a_1; a_2, \dots].$$

In the same way, we can write

$$u' = a_1 + \frac{1}{[a_2; a_3, \dots]},$$

and so $u' \in (a_1, a_1 + 1)$, showing that u', and hence u uniquely determines a_1. Inductively, by repeating the same argument again and again, we may find that u uniquely determines all the terms in the continued fraction.

P2. Let $u = [0; a_1, a_2, \dots] \in (0, 1)$, and let $\frac{p_n}{q_n}$ be its n-th convergent. Show that for any $n \in \mathbb{N}$

$$\left| u - \frac{p_n}{q_n} \right| > \frac{1}{q_n q_{n+2}}.$$

Since $\frac{p_n}{q_n} < u < \frac{p_{n+1}}{q_{n+1}}$ or $\frac{p_{n+1}}{q_{n+1}} < u < \frac{p_n}{q_n}$, it follows that

$$\left| u - \frac{p_n}{q_n} \right| = \left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \right| - \left| \frac{p_{n+1}}{q_{n+1}} - u \right|.$$

Thus, by Proposition $72(ii)$ from the lecture notes, we have that

$$\left| \frac{p_{n+1}}{q_{n+1}} - u \right| < \frac{1}{q_{n+1} q_{n+2}},$$

and

$$\left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \right| = \left| \frac{p_{n+1} q_n - p_n q_{n+1}}{q_{n+1} q_n} \right| = \frac{1}{q_{n+1} q_n}.$$

Therefore

$$\left| u - \frac{p_n}{q_n} \right| > \frac{1}{q_{n+1} q_n} - \frac{1}{q_{n+1} q_{n+2}} = \frac{q_{n+2} - q_n}{q_n q_{n+1} q_{n+2}} = \frac{a_{n+2}}{q_n q_{n+2}} \geq \frac{1}{q_n q_{n+2}},$$

where we used that $q_{n+2} = a_{n+2}q_{n+1} + q_n$, again from Proposition 72.

P3. A real number $u = [0; a_1, a_2, \ldots] \in (0,1)$ is called *badly approximable* if there exists some $M \in \mathbb{R}$ such that $a_n \leq M$ for all $n \geq 1$.

Show that $u \in (0,1)$ is badly approximable if and only if there exists some $\varepsilon > 0$ such that

$$\left| u - \frac{p}{q} \right| \geq \frac{\varepsilon}{q^2}$$

for all rational numbers $\frac{p}{q}$.

($\Longrightarrow$) Assume that $u \in (0,1)$ is badly approximable. Write $u = [0; a_1, a_2, \ldots]$, and let $\frac{p_n}{q_n}$ be its n-th convergent. Recall, by Proposition 72 (of lecture notes), that $q_{n+1} = a_{n+1}q_n + q_{n-1}$, and thus by hypothesis

$$q_{n+1} \leq (M+1)q_n,$$

using that $q_n \geq q_{n-1}$. Now, fix $n \in \mathbb{N}$ such that $q \in (q_{n-1}, q_n]$. We have that

$$\left| \frac{p}{q} - u \right| > \left| \frac{p_n}{q_n} - u \right|,$$

by Theorem 76 and definition of the best approximate. Thus, by the previous problem, we have

$$\left| \frac{p}{q} - u \right| > \left| \frac{p_n}{q_n} - u \right| > \frac{1}{q_n q_{n+2}} \geq \frac{1}{(M+1)^2 q_n^2} \geq \frac{1}{(M+1)^4 q_{n-1}^2} \geq \frac{1}{(M+1)^4 q^2}.$$

($\Longleftarrow$) Conversely, assume that for some $\varepsilon > 0$,

$$\left| u - \frac{p}{q} \right| \geq \frac{\varepsilon}{q^2},$$

for all rational $\frac{p}{q}$. In particular, for all $n \geq 1$,

$$\frac{\varepsilon}{q_n^2} \leq \left| u - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}},$$

where the last inequality comes from Proposition 72 of the lecture notes. In particular

$$\varepsilon^{-1} > \frac{q_{n+1}}{q_n} = \frac{a_{n+1}q_n + q_{n-1}}{q_n} = a_{n+1} + \frac{q_{n-1}}{q_n} \geq a_{n+1},$$

for all $n \geq 2$. Taking $M \geq \text{máx}\{\varepsilon^{-1}, a_1, a_2\}$, we conclude.

P4. Let $(X, \mathscr{B}, \mu, T)$ be an ergodic measure-preserving system and let A be a set of positive measure. The pointwise ergodic theorem shows that for almost all $x \in X$, the set of visiting times

$$\Lambda_x = \{n \in \mathbb{N} \colon T^n x \in A\}$$

has natural density equal to $\mu(A)$ (Corollary 65(iii)). Is it true that, for almost all $x \in X$, the set Λ_x has bounded gaps, i.e. it is *syndetic*?

> The answer is negative. We provide an example where, for almost all x, the set Λ_x does not have bounded gaps.
>
> Let $(\mathbb{T}, \mathscr{B}(T), \mu)$ be the circle ($\mathbb{T} = \mathbb{R}/\mathbb{Z}$) with the Borel σ-algebra and the Lebesgue measure and consider the map $Tx = 10x \pmod 1$. Consider the set $A = [0.4.0.5]$ which has positive measure. Note that $\{10^n x\} \in A \iff a_{n+1} = 4$.
>
> By Borel's Theorem on normal numbers, we know that almost all $x \in \mathbb{T}$ are normal in base 10 and thus attain all patterns of digits infinitely often. Take a normal number x, let $a_n(x)$ denote its n-th digit in base 10 and assume that $\Lambda_x = \{n \in \mathbb{N} \colon \{10^n x\} \in A\}$ has gaps bounded by K (we allow K to depend on x). Consider the digit pattern $(1, \ldots, 1)$ consisting of $2K+1$ consecutive 1s. Then, we know from the definition of normality that we have $a_n(x) = a_{n+1}(x) = \ldots = a_{n+2K-1}(x) = a_{n+2K}(x) = 1$ for infinitely many n. Therefore, $\{10^m x\} \leq 0.2$ for all $m \in \{n, n+1, \ldots n+2K-1\}$. In particular, $\{10^m x\} \notin A$ for all m in this range, contradicting the fact that Λ_x has gaps bounded by K.
>
> An additional example: Let $(X, \mathscr{B}(X), \mu)$, where $X = \{0,1\}^{\mathbb{N}}$ endowed with the Borel σ-algebra $\mathscr{B}(X)$ and the product probability measure μ for some fixed $p \in (0,1)$. Consider the left-shift map T and take the set $A = \{x \in X \colon x_0 = 1\}$, for which $\mu(A) = p$. One can check that for almost every x, Λ_x has unbounded gaps.